

Exact Electric-Magnetic Duality in $N = 2$ Supersymmetric QCD Theories

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We analyze the Coulomb phase of theories of $N = 2$ SQCD with $SU(N_c)$ gauge groups which are conjectured to have exact electric-magnetic duality. We discuss the duality transformation of the particle spectrum, emphasizing the differences between the general case and the $SU(2)$ case. Some difficulties associated with the definition of the duality transformation for a general gauge group are discussed. We compute the classical monopole spectrum of these theories, and when it is possible we use it to check the consistency of the duality. Generally these theories may have phase transitions between strong and weak coupling, which prevent the semi-classical computation from being useful for checking the duality.

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1. Introduction

Electric-magnetic duality [1] has played a major role in the exciting recent developments in our understanding of gauge theories and string theories. Generally, this duality is not an exact symmetry, but only a transformation between different descriptions of the low-energy (IR) theory. However, there are two classes of theories which are conjectured to have an exact electric-magnetic duality symmetry. These are the $N = 4$ SYM theories [2], and $N = 2$ SQCD theories with a matter content chosen such that the beta function vanishes perturbatively [3], and is conjectured to vanish also non-perturbatively [4,5]. Both of these theories are also conjectured to be finite. The electric-magnetic duality in other $N = 2$ SQCD theories may be derived by flowing from the scale invariant theories [5], and some sort of flow may also relate Seiberg's $N = 1$ duality [6] to the duality in scale invariant $N = 2$ theories [7].

For $N = 4$ SYM theories, the transformation of the $SL(2, \mathbf{Z})$ duality on the particle spectrum for general gauge groups is known since the late seventies [1]. The electric charges in this case sit in the root lattice of the gauge group, $\vec{e} = \sum_i n_e^i \vec{\alpha}^i$, where the sum is over the simple roots of the gauge group G . The magnetic charges sit in the root lattice of the dual gauge group, $\vec{g} = \sum_i n_m^i \frac{\vec{\alpha}^i}{(\vec{\alpha}^i)^2}$. The $SL(2, \mathbf{Z})$ duality acts on the (non-running) coupling constant $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ of the high-energy gauge group G , and is generated by $T : \tau \rightarrow \tau + 1$ and by $S : \tau \rightarrow -\frac{1}{\tau}$. The generator S exchanges electric and magnetic charges, and exchanges the roots with the dual roots : $\vec{\alpha}^i \leftrightarrow \frac{\vec{\alpha}^i}{(\vec{\alpha}^i)^2}$, up to a constant depending on the normalization chosen for the roots. The gauge boson with electric charge $\vec{\alpha}$ is thus transferred by S to the magnetic monopole with magnetic charge $\frac{\vec{\alpha}}{\vec{\alpha}^2}$, which is exactly the embedding of the 't Hooft Polyakov monopole in the direction of the root $\vec{\alpha}$ [8]. This transformation exchanges the group G with the dual gauge group G^\vee . For $SU(N_c)$ the dual group is $SU(N_c)/\mathbf{Z}_{N_c}$, while for groups having roots of unequal length, the dual group generally has a different algebra than the original algebra. The verifications of this duality so far have involved the curve describing the low-energy $U(1)^r$ gauge theory [5,9], and various partition functions [10,11], all of which were found to be $SL(2, \mathbf{Z})$ invariant. Another verification of the duality comes from checking that the BPS-saturated particle spectrum of the theory is invariant under the duality. For $N = 4$ SYM, the only BPS states which are only electrically charged are the gauge bosons, whose charge \vec{e} goes over all roots of the gauge group. The tests made so far on the duality of the spectrum have involved finding the $SL(2, \mathbf{Z})$ duals of these states by a semi-classical analysis. As discussed below,

the semi-classical computation is sufficient for the $N = 4$ SYM theory, since the spectrum is the same at strong and weak coupling. For gauge group $SU(2)$ the BPS spectrum has been shown to be consistent with the duality, for magnetic charge 2 by Sen [12], and for all magnetic charges by Porrati [13]. For higher gauge groups it is clear that the S -duals of the W bosons exist semi-classically, since these are just the embeddings of the 't Hooft Polyakov monopole, but it has not yet been shown that the complete spectrum is indeed dual (for instance, that no other magnetic monopoles with zero electric charge exist).

$N = 2$ SQCD theories with vanishing beta function are known to be superconformally invariant and finite at the perturbative level [3], and they are conjectured (as we will assume from here on) to be superconformally invariant and finite also non-perturbatively. Seiberg and Witten have conjectured that these theories also have an exact electric-magnetic duality for $SU(2)$ gauge group, and this has since been generalized to higher gauge groups. However, so far the only verification of the duality for higher gauge groups is in the spectrum-generating curve of the low energy $U(1)^r$ theory, which is invariant under a subgroup of $SL(2, \mathbf{Z})$. Since massless particles cause monodromies that may be read from the curve, the invariance of the curve implies that the particles which become massless anywhere in the moduli space are also invariant under the corresponding duality subgroup. However, apriori it says nothing about particles that are massive in all of moduli space, or about particles that are massless only together with other particles which are non-locally related to them [14,15]. In this case it is not obvious whether the quantum numbers of the massless particles may be read from the curve. In the $SU(2)$ case, the duality was also verified by checking that the BPS-saturated particle spectrum is $SL(2, \mathbf{Z})$ invariant for all states which have been computed semi-classically [5,16]. In this paper we wish to generalize this to higher gauge groups. We will discuss only $SU(N_c)$ groups, but we expect similar results to be true for all gauge groups.

Let us start by reviewing the situation for gauge group $SU(2)$ [4,5]. In this case, Seiberg and Witten conjectured, for $N_f = 4$ (where N_f is the number of hypermultiplets in the fundamental representation), the existence of an $SL(2, \mathbf{Z})$ duality symmetry acting on $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ (note the factor of two difference from the previous case). This time there are two kinds of states having only electric charge : the quarks with charges $(n_m, n_e) = (0, \pm 1)$, in the vector representation of the $SO(8)$ flavor group, and the W bosons with charges $(0, \pm 2)$ which are flavor singlets. The $SL(2, \mathbf{Z})$ duality transformations were found to involve also outer automorphisms of the $SO(8)$ flavor group. Since all non-singular points in the quantum moduli space are smoothly connected to the semi-classical weak coupling

region, a semi-classical computation of the BPS spectrum is expected to give the correct result throughout the moduli space. The spectrum of states with $n_m = 1, 2$ was analyzed semi-classically by several groups [16], and was found to be consistent with the duality. All states which can be reached by a duality transformation on the elementary states were indeed found, and, for these values of n_m , no other states were found. For general n_m the semi-classical analysis of the spectrum involves analyzing the zero modes of a certain operator on the moduli space of n_m monopoles, which is a complicated hyperKähler manifold. This analysis has not yet been performed.

For general gauge groups much less is known about the $N = 2$ duality. For all simple non-exceptional gauge groups, the curve whose periods generate the spectrum has been found by now for all interesting values of N_f [17,18,19,20]. The results are summarized in the introduction of [19]. The parameters of this curve in the scale invariant cases are the gauge coupling τ and gauge invariant polynomials built from the Higgs field Φ , all of which are uniquely defined only for weak coupling (different definitions may differ for strong coupling). The gauge symmetry is generically broken to $U(1)^r$ (where r is the rank of the gauge group), and the gauge couplings are given by a matrix $\tau_{ij} = \partial_i \partial_j \mathcal{F}$. At the classical level τ_{ij} is just a constant matrix, proportional to the Cartan matrix of the group when the $U(1)$ factors are defined to be aligned with the simple roots of the gauge group. In the case of $SU(3)$ gauge group with $N_f = 6$, on a non-singular subsurface of the moduli space, this is true also at the quantum level, as described below. The low energy $U(1)^r$ gauge theory has an $Sp(2r, \mathbf{Z})$ transformation group, generated by S transformations on each $U(1)$ factor separately, by shifts of the various theta angles and by rotations mixing the various $U(1)$ groups. A subgroup of this transformation group keeps the matrix τ_{ij} proportional to the Cartan matrix (in the basis mentioned above), and a subgroup of this group may be an exact symmetry of the scale-invariant theory. For gauge groups other than $SU(2)$, this subgroup turns out not to be $SL(2, \mathbf{Z})$. This is revealed by looking at the curves and seeing which transformations of the coupling constant leave them invariant. Instead, it is a subgroup of $SL(2, \mathbf{Z})$ which depends on the gauge group. For $SU(N_c)$ ($N_c > 2$) and $SO(N_c)$ ($N_c > 4$), in the parametrization of the curves given in [19], the duality subgroup acting on the coupling is generated by S and by T^2 . For $Sp(N_c)$ it is generated by T and by ST^2S . Note that T^2 is now the shift of the theta angle by 2π , and that a shift of the theta angle by π is not generally a symmetry.

In this paper we study the action of the duality transformations on the particle spectrum of the $N = 2$ SQCD theories, in an attempt to verify the existence of an exact duality

symmetry for all gauge groups. We begin in section 2 with a general description of the $N = 2$ SQCD theories for general gauge groups. In section 3 we check which $Sp(2r, \mathbf{Z})$ transformations preserve the form of the classical coupling matrix τ_{ij} . For $SU(3)$ gauge group the classical coupling matrix is exact on a subsurface in moduli space. In section 4 we compute the classical monopole spectrum of the $N = 2$ theory, finding the number of bosonic zero modes around any classical monopole solution. In section 5 we add also the fermionic zero modes, and describe the quantum numbers of the semi-classical monopoles. In section 6 we check if the semi-classical spectrum is consistent with electric-magnetic duality. Unfortunately, we find that the semi-classical verification is only possible in a part of moduli space in which we were not able to complete the full semi-classical analysis (due to mathematical difficulties). We show, however, that the semi-classical spectrum could be consistent with the duality. In section 7 we analyze the theories with non-zero bare quark masses, and see that also for these theories we cannot verify or rule out the duality by a semi-classical analysis. We end in section 8 with a summary and conclusions.

2. General description of $N = 2$ SQCD

We consider $N = 2$ supersymmetric gauge theories with a gauge group $G = SU(N_c)$, and with N_f hypermultiplets in the fundamental representation (‘quarks’), with N_f chosen so that the beta function of the theory vanishes perturbatively (for $SU(N_c)$ gauge groups $N_f = 2N_c$). The field content of these theories includes the $N = 2$ vector multiplet, whose scalar component will be denoted by ϕ . In terms of $N = 1$ superfields the $N = 2$ vector multiplet consists of a vector superfield W_α and a chiral superfield Φ , both in the adjoint representation of the gauge group. The N_f $N = 2$ hypermultiplets consist of two chiral superfields, Q_a^i and \tilde{Q}_i^a , where $i = 1, \dots, N_f$ is a flavor index and $a = 1, \dots, N_c$ is a color index. The superpotential in the $N = 1$ language, for zero quark masses (as we will assume until section 7), is

$$W = \sqrt{2} \tilde{Q}_i \Phi Q^i, \quad (2.1)$$

suppressing the color indices. The flavor symmetry of this theory (for $N_c > 2$) is $SU(N_f) \times U(1)_B$. The global symmetry includes also the usual $SU(2)_R \times U(1)_R$ factors of the classical $N = 2$ theory. For $N_f = 2N_c$, there is no perturbative anomaly in the $U(1)_R$ symmetry, and we will assume that it remains unbroken in the full quantum theory (except for spontaneous breaking).

We will be interested here only in the Coulomb phase of these theories, in which the only field obtaining a VEV is ϕ . The equations of motion imply $[\phi, \phi^\dagger] = 0$, so that in a vacuum we can always diagonalize the matrix $\langle \phi \rangle$, i.e. choose it to be in the Cartan subalgebra of the gauge group. Choosing the basis of the Cartan subalgebra to be the simple roots of the gauge group, we can thus take

$$\langle \phi \rangle = \sum_{i=1}^r a_i \vec{\alpha}^i \quad (2.2)$$

where r is the rank of the gauge group. Classically, the moduli space is labeled by the value of the a_i up to gauge transformations. After choosing $\langle \phi \rangle$ in the Cartan subalgebra, we are left only with the freedom to perform Weyl transformations.

For $SU(N_c)$ gauge groups the Coulomb phase moduli space of vacua may be parametrized by the gauge invariant operators $u_k = \frac{1}{k} \langle \text{Tr}(\Phi^k) \rangle$ for $k = 2, \dots, N_c$. Equivalently, one may use s_k , the symmetric polynomials in the eigenvalues of Φ . Classically, these determine the a_i up to Weyl transformations. In the theories we analyze we expect to have exact scale invariance and exact $U(1)_R$ invariance. Hence, the particle spectrum at the point $\{u_k\}$ in the moduli space should be the same as the particle spectrum at the point $\{\lambda^k u_k\}$ for any complex λ . Dimensionless parameters, such as the effective gauge coupling, may still depend (for $N_c > 2$) on dimensionless ratios of the u_k , such as u_3^2/u_2^3 . Unlike the case of gauge group $SU(2)$, for higher rank groups the $U(1)_R$ symmetry does not enable us to choose $\langle \phi \rangle$ to be real. At best, we can (and will) choose $\langle \phi \rangle \cdot \vec{\alpha}$ to be real for some particular $\vec{\alpha}$ in the Cartan subalgebra.

At a generic point in moduli space, the VEV of ϕ breaks the gauge group to $U(1)^r$, and a low energy effective theory may be written in terms of $U(1)$ vector multiplets (A_i, W_i) ($i = 1, \dots, r$), which are the only massless fields. We can choose the basis of the $U(1)^r$ gauge group such that the VEV of the scalar component of A_i is exactly a_i defined above. The $N = 2$ effective lagrangian takes the form

$$L_{eff} = Im \frac{1}{4\pi} \left[\int d^4\theta \partial_i \mathcal{F}(A) \bar{A}^i + \frac{1}{2} \int d^2\theta \partial_i \partial_j \mathcal{F}(A) W^i W^j \right] \quad (2.3)$$

where \mathcal{F} is a holomorphic prepotential [21]. Classically, in the basis we chose, \mathcal{F} is proportional to

$$\mathcal{F}_{cl}(A) \propto \tau \sum_{i,j} A_i A_j C_0^{ij} \quad (2.4)$$

where $\tau = \theta/\pi + 8\pi i/g^2$ is the bare gauge coupling (which is well defined for $N_f = 2N_c$) and C_0 is the Cartan matrix of the gauge group ($C_0^{ij} = \frac{2\vec{\alpha}^i \cdot \vec{\alpha}^j}{(\vec{\alpha}^j)^2}$). We will choose to normalize the coupling so that θ is the coefficient of the topological term in the lagrangian³.

When $u_i = 0$ for $i = 2, \dots, N_c - 1$, and only u_{N_c} is non-zero, the one-loop correction to the low energy coupling $\tau_{ij} = \partial_i \partial_j \mathcal{F}$ is a constant matrix⁴, essentially because there is only one scale in the theory, and the masses of all particles are multiples of this scale. For $G = SU(3)$ the constant one-loop correction vanishes, and the classical result is perturbatively exact on this subsurface. Other surfaces with only one scale also exist, but on all of them, except this one, we have some massless quarks, and then the effective coupling runs below the scale of the massive particles. We will denote this subsurface of the moduli space by \mathcal{M} . In the $N = 4$ theory, the one-loop correction vanishes throughout moduli space, due to the cancelation between the $N = 2$ vector multiplet and hypermultiplet, but this is not generally true in the $N = 2$ theory. The higher perturbative corrections always vanish. For $SU(2)$ gauge group there are no non-perturbative corrections to this expression in the scale invariant theories. It is reasonable to expect that this will be true also for larger gauge groups. For $G = SU(3)$ on the subsurface \mathcal{M} , the expression for the dual variables a_D^i is then exactly given by

$$a_D^i = \partial_i \mathcal{F} = K \tau C_0^{ij} a_j \quad (2.5)$$

where K is a constant. For the $N = 4$ theories, (2.4) and (2.5) are exact throughout the whole moduli space.

The states of this theory generally carry electric and magnetic charges. The electric charges live in the weight lattice of the gauge group. With an appropriate normalization, we can write $\vec{e} = \sum_{i=1}^r n_e^i \vec{\mu}_i$, where the weights $\vec{\mu}_i$ are defined by $\vec{\mu}_i \cdot \vec{\alpha}^j = \frac{1}{2} \delta_i^j (\vec{\alpha}^j)^2$ and the charges n_e^i are all integers. The magnetic charges \vec{g} (appropriately normalized) must then satisfy the Dirac quantization condition $\vec{e} \cdot \vec{g} = 2\pi n$ for all \vec{e} of purely electric states. This implies $\vec{g} = 4\pi \sum_{i=1}^r n_m^i \vec{\beta}^i$, where $\vec{\beta}^i = \frac{\vec{\alpha}^i}{(\vec{\alpha}^i)^2}$ and the n_m^i are all integers. Thus, the magnetic charges lie in the root lattice of the dual gauge group (we will normalize \vec{g} without the 4π factor from here on to simplify the notations). Note that this quantization is different than the case of an unbroken simple gauge group, when the magnetic charges

³ This convention differs by a factor of N_c from the convention used when describing the large N_c limit of these theories.

⁴ This point has been stressed in a recent paper [32].

may lie in the weight lattice of the dual gauge group. The mass of a state is bounded by the BPS bound, which for zero quark masses is given by

$$M \geq \sqrt{2}|Z| = \sqrt{2}|n_e^i a_i + n_m^i a_D^i|, \quad (2.6)$$

with equality only for states in small representations of the $N = 2$ algebra.

The low energy action has an $Sp(2r, \mathbf{Z})$ group of transformations, acting on (a, a_D) (now viewed as vectors of r elements) by

$$\begin{pmatrix} a \\ a_D \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ a_D \end{pmatrix} \quad (2.7)$$

and on (n_e, n_m) in an appropriate way so that Z of (2.6) is invariant. The matrix of gauge couplings $\tau_{ij} = \partial_i \partial_j \mathcal{F}$ is transformed by (2.7) as $\tau \rightarrow (C + D\tau)(A + B\tau)^{-1}$. In general these transformations are not expected to be exact symmetries of the $N = 2$ SQCD theory, but some subgroup of $Sp(2r, \mathbf{Z})$ may in fact be an exact symmetry, as discussed in the next section. Such a symmetry should preserve the matrix form of τ_{ij} , changing only the gauge coupling coefficient.

An important difference which arises when the rank of the gauge group is larger than one, is that even in the scale invariant theories we have singular surfaces of real codimension one on which states of the theory are only marginally stable. These occur when the central charges Z corresponding to two different states of the theory have the same complex phase. In addition to these surfaces, there exist singular surfaces of real codimension two, corresponding to massless fields, which can be either massless vectors (which occur for instance when the gauge symmetry is not completely broken to an abelian subgroup) or massless hypermultiplets. The codimension two singular surfaces generate $Sp(2r, \mathbf{Z})$ monodromies when going around them. On the other hand, the spectrum of the theory can jump upon crossing codimension one singular surfaces on which states are only marginally stable [23]. States which are only marginally stable on the singular surface may exist as stable particles on one side of the surface but not on the other. This happens for instance in the $SU(2)$ case, even for $N_f = 0$ [4]. As we will see in sections 4 and 5, the semi-classical computation of the spectrum indicates that such jumps do indeed occur even in the semi-classical region.

In the $SU(2)$ theory with $N_f = 4$ [5], we do not cross any singular surfaces of this type when the coupling is changed from strong coupling to weak coupling, because the expression $a_D = \tau a$ is exact. In the $N = 2$ theories of $SU(N_c)$ for $N_c > 2$, due to quantum

corrections, this is no longer the case at a generic point in moduli space. Hence, the particle spectrum at weak coupling, which we can compute by a semi-classical analysis, is not necessarily the same as the spectrum at strong coupling. From the known form of the low-energy curve given below, we can see that there are no codimension one singular surfaces which intersect the subsurface \mathcal{M} . However, for $N_c > 3$ we cannot show that there are no codimension two singular surfaces which intersect \mathcal{M} , so we cannot be sure that the spectrum does not change when we go from weak to strong coupling. Only for $SU(3)$, the relation (2.5) is exact along the surface \mathcal{M} . Therefore, along this surface, the spectrum should be the same at strong and weak coupling, and its semi-classical computation should be self-dual. We will only be able to perform semi-classical tests of the duality near this special surface in moduli space. In principle, the exact form of the singular surfaces may be computed from the low-energy curves, and then we can tell exactly where the semi-classical calculation is reliable and where it is not. Since we have not done this, we will have to restrict ourselves to rigorously checking the duality only for $SU(3)$ near the surface \mathcal{M} , where we know that the spectrum is the same at strong and weak coupling. For $N_c > 3$ we cannot prove that there is a region of moduli space in which no phase transitions occur. However, we will see below that the spectrum along \mathcal{M} is consistent with the duality. Thus, it is possible that no phase transitions occur along this surface. Note that for the $N = 4$ theories, there are no quantum corrections and (2.5) is valid throughout the moduli space. Therefore, in these theories the spectrum is the same at strong and weak coupling for all gauge groups, and the semi-classical spectrum should be self-dual.

3. Duality transformations of $N = 2$ SQCD

Several groups have constructed curves describing the low-energy physics in the Coulomb phase of $N = 2$ SQCD theories for general gauge groups. For $SU(N_c)$ gauge groups with $N_f = 2N_c$, the curve (in the parametrization of [18]) is given by

$$y^2 = \langle \det(x - \Phi) \rangle^2 + 4h(\tau)(h(\tau) + 1) \prod_{j=1}^{2N_c} (x - m_j - \frac{1}{N_c} h(\tau) \sum_{k=1}^{2N_c} m_k), \quad (3.1)$$

where the function $h(\tau)$ is given by $h(\tau) = \frac{\theta_2^4(\tau)}{\theta_4^4(\tau) - \theta_2^4(\tau)}$ (for weak coupling $h(\tau) \sim 16e^{i\pi\tau}$), and the m_k are the masses of the quarks. The parameter τ appearing in the curve is related to the (non-running) gauge coupling of the high-energy non-abelian gauge group,

and should be equal to it at least for weak coupling. This curve has two symmetry transformations which leave it invariant. One is the transformation $T^2 : \tau \rightarrow \tau + 2$, corresponding to $\theta \rightarrow \theta + 2\pi$ which is obviously a symmetry of the theory. Another symmetry transformation is $S : \tau \rightarrow -1/\tau$, which is assumed to be related to electric-magnetic duality. The S transformation also inverts the sign of the singlet quark mass but not of the $SU(N_f)$ -adjoint masses, corresponding to an outer automorphism of the flavor group which inverts the $U(1)_B$ charge and does not act on $SU(N_f)$. Both of these symmetry transformations do not act on the gauge invariant variables u_k (appearing in the determinant in (3.1)), which are the other parameters of the curve.

The relation between the parameter τ of the curve and the actual gauge coupling is complicated, and is not a one-to-one transformation. This can easily be seen by comparing the parametrization (3.1) with other parametrizations of the curve. For $SU(3)$ another parametrization was given in [22], which had duality transformations S and T which satisfied $(ST)^6 = I$, while no such relation is satisfied by the symmetry generators of (3.1)⁵. For $SU(2)$, we can compare the S transformation above to the transformations given by Seiberg and Witten [5], and see that S above corresponds in their parametrization to a transformation of the form TST , which does not square to unity when acting on the coupling defined in [5]. The $SU(2)$ example also clearly shows that the transformations which are easily read from (3.1) are not necessarily the most general duality transformations. For instance, the S transformation of [5] is not easily visible in the parametrization (3.1) of the curve. We will, therefore, look in this paper for more general possible forms of duality transformations. In general, these dualities could also involve different outer automorphisms of the $SU(2N_c) \times U(1)_B$ flavor group. The $SU(2N_c)$ group has just one non-trivial outer automorphism, which conjugates the $SU(2N_c)$ representations, but we could also have inversions of the baryon number charge, and shifts of the baryon number charge by the \mathbf{Z}_{2N_c} charge corresponding to the center of the $SU(2N_c)$ group. In general, the duality transformations may also act on the u_k , but we will look only for transformations which leave the u_k invariant.

We consider now general electric-magnetic duality transformations, which transform the $N = 2$ SQCD theory with high energy coupling τ to the same theory with high energy coupling $\tau' = f(\tau)$. The respective low energy theories should be related by an

⁵ Another way to explain this would be if the S transformation of [22] is not easily visible in the parametrization (3.1) and vice versa.

$Sp(2r, \mathbf{Z})$ transformation. We begin by assuming that the classical relation (2.5) holds both before and after the duality transformation. As discussed above, this is true also quantum mechanically for the cases of $G = SU(2)$, $G = SU(3)$ along the surface \mathcal{M} ⁶ and $N = 4$ SYM. For the other cases the quantum corrections are important and will be discussed at the end of this section.

With this assumption, we should look for $Sp(2r, \mathbf{Z})$ transformations which leave the relation (2.5) invariant, up to a possible change of τ . Equivalently, they should preserve the matrix form of the coupling matrix $\tau_{ij} = K\tau C_0^{ij}$. Let us now take a general $Sp(2r, \mathbf{Z})$ transformation, and a general transformation $\tau \rightarrow f(\tau)$, and check which transformations leave the relation (2.5) invariant. We will work in matrix notation, as in equation (2.7). After the transformation, a_D is given by $Ca + Da_D$ and a is given by $Aa + Ba_D$. Equation (2.5) then becomes

$$Ca + Da_D = Kf(\tau)C_0(Aa + Ba_D). \quad (3.2)$$

Plugging in the relation (2.5) for the original variables, we find the matrix equation

$$C + K\tau DC_0 = Kf(\tau)C_0(A + K\tau BC_0) \quad (3.3)$$

which must be satisfied for all τ .

Since $f(\tau)$ should be a holomorphic function, we can take derivatives of (3.3) with respect to τ . By demanding equality for all τ we then find that all matrices appearing in (3.3) (C, DC_0, C_0A and C_0BC_0) must be proportional to each other, and $f(\tau)$ must be of the form $f(\tau) = (a\tau + b)/(c\tau + d)$. The fact that all these matrices are proportional, means that if we start from a state whose magnetic and electric charges (the vectors \vec{g} and \vec{e} defined above) are proportional to each other, we will remain with proportional electric and magnetic charges. This results from the fact that C_0 is proportional, for simply laced groups, to the matrix transforming the electric charge basis to the magnetic charge basis. In particular, states which are charged only electrically or only magnetically are necessarily transformed into states whose electric charge is in the same direction as their magnetic charge.

Let us analyze first the case in which B is zero. Then equation (3.3) becomes

$$C + K\tau DC_0 = Kf(\tau)C_0A. \quad (3.4)$$

⁶ Since we assume that the duality transformation does not change the u_k , it leaves the surface \mathcal{M} invariant.

Clearly, the only possible solutions to this equation for all τ are of the form $f(\tau) = a_0\tau + b_0$. Since $A^T D = I$ in this case (where I is the identity matrix), we can easily show that $a_0 = \pm 1$, and obviously only $a_0 = 1$ is physically relevant, since b_0 is real and the imaginary part of τ is always non-negative. These transformations are exactly the transformations of the form T^{2n} described above, and we see that they can indeed leave the theory invariant. Since $T^2 : \tau \rightarrow \tau + 2$ is supposed to take $\vec{e} \rightarrow \vec{e} + \vec{g}$ (when the length of the roots is normalized to one), we find that $K = 1/2$ for all N_c in our conventions. Obviously these transformations do not exchange electric and magnetic charges; for electric-magnetic duality we must obviously have non-zero B which is the next case we shall analyze.

Since we found that all matrices in (3.3) are proportional, we can write (for non-zero B) $A = q_1 B C_0$, $C = q_3 C_0 B C_0$, and $D = q_4 C_0 B$, where q_1, q_3 and q_4 are constants, which must obviously be rational numbers, since all matrices are integer valued. The transformation on τ is then

$$\tau \rightarrow f(\tau) = \frac{q_3 + K q_4 \tau}{K(q_1 + K \tau)}. \quad (3.5)$$

Using the $Sp(2r, \mathbf{Z})$ relation $A^T D - C^T B = I$ we can see that

$$(q_1 q_4 - q_3) C_0 B^T C_0 B = I. \quad (3.6)$$

This equation essentially means that B should preserve the form of the lattice of charges as a sublattice of the Cartan subalgebra. Taking the determinant of this equation we find that

$$(q_1 q_4 - q_3)^r = 1/(\det(C_0) \det(B))^2. \quad (3.7)$$

By analogy with the $SU(2)$ case, we will first look for solutions in which $A = D = 0$, i.e. there is no mixing between electric and magnetic charges. In this case, taking the determinant of $C = q_3 C_0 B C_0$, we find that $\det(C) = (-1)^r / \det(B)$, but both determinants must be integers, hence $\det(B) = \pm 1$. Plugging this into (3.7), we find that there are no rational solutions for q_3 for $SU(N_c)$ groups with $N_c > 3$ (recall that $\det(C_0) = N_c$). Thus, for these groups, any duality transformation must mix electric and magnetic charges in a more complicated way than for $SU(2)$. For generic N_c the smallest value of $\det(B)$ for which (3.7) has a rational solution is $\det(B) = \pm N_c^{N_c-2}$, and then $q_1 q_4 - q_3 = 1/N_c^2$. We can always choose, for instance, $B = N_c C_0^{-1}$ (which is an integer matrix) and satisfy all the equations. For particular values of N_c smaller values of $\det(B)$ are also possible. All solutions of (3.6) give rise to transformations which preserve the low energy effective action

and the form of the low energy couplings. To find which of the solutions are exact symmetries of the theory we must go beyond the low energy action, for instance by studying the particle spectrum of the theories, which we do in the next sections.

The duality transformation of the charges is the inverse of the transformation on (a, a_D) . For the transformation matrices given above we find that

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \rightarrow \begin{pmatrix} q_4 C_0 B & -q_3 C_0 B C_0 \\ -B & q_1 B C_0 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}. \quad (3.8)$$

Denoting $\vec{g}_i = -B_{ji}\vec{\beta}_j$, and normalizing all root lengths to unity (which is possible for simply laced groups), we find that the transformation of the basis of the charge lattice is given by

$$\begin{aligned} \vec{e} = \vec{\mu}_i &\rightarrow \vec{e} = -q_4 \vec{g}_i & \vec{g} = \vec{g}_i \\ \vec{g} = \vec{\beta}_i &\rightarrow \vec{e} = q_3 (C_0)_{ji} \vec{g}_j & \vec{g} = -q_1 (C_0)_{ji} \vec{g}_j. \end{aligned} \quad (3.9)$$

Using (3.6) it is easy to verify that this transformation preserves the symplectic product $\vec{e}_1 \cdot \vec{g}_2 - \vec{g}_1 \cdot \vec{e}_2$ between any pair of vectors. The freedom to perform $T^2 : \tau \rightarrow \tau + 2$ transformations corresponds here to a freedom to transform the rational numbers q_i by $q_3 \rightarrow q_3 + 2Kq_4; q_1 \rightarrow q_1 + 2K; q_4 \rightarrow q_4$ or by $q_3 \rightarrow q_3 + 2Kq_1; q_4 \rightarrow q_4 + 2K; q_1 \rightarrow q_1$. Both transformations of course preserve $(q_1 q_4 - q_3)$. Thus, up to T^2 transformations, we can always choose the absolute values of q_1 and q_4 to be no larger than $\frac{1}{2}$.

Up to now we did not add any constraints on the duality from the flavor quantum numbers of the various states. As we discuss in section 6, for all states in the theory the charge (n -ality) under the center of the $SU(2N_c)$ flavor group is equal (modulo N_c) to the charge under the center of the $SU(N_c)$ gauge group (given by $\sum_i in_e^i$). For $N_c > 2$, since there is only one $SU(2N_c)$ representation (up to conjugation) of size $2N_c$, and the duality preserves the number of states, it seems that the duality must also preserve the $SU(N_c)$ n -ality of states, or invert it if an $SU(2N_c)$ outer automorphism is also involved. From (3.9) we can easily see that for this to happen, q_4 cannot be an integer, since then all weights would transform to roots, whose n -ality is zero. In fact q_4 must be of the form p/q where q is an integer multiple of N_c , and this will constrain the possible transformations. We assumed here that the monopoles carry no charge under the center of the gauge group – if this is not correct then we should add also the n -ality of the monopoles to this discussion.

The transformation matrix $B = N_c C_0^{-1}$ seems to be the most natural choice. In this case, the transformation is similar to the $N = 4$ transformation, taking electric charge vectors to magnetic charge vectors which are proportional to them (as vectors in the

Cartan subalgebra). For this transformation we find $q_1 q_4 - q_3 = 1/N_c^2$, to which the simplest solution giving an $Sp(2r, \mathbf{Z})$ transformation which preserves or inverts the n -ality of electric charge vectors is obviously $q_1 = q_4 = \pm 1/N_c, q_3 = 0$. Equation (3.9) in this case becomes

$$\begin{aligned} \vec{e} = \vec{\mu}_i &\rightarrow \vec{e} = \mp \vec{\mu}_i & \vec{g} &= N_c \vec{\mu}_i \\ \vec{g} = \vec{\beta}_i &\rightarrow \vec{e} = 0 & \vec{g} &= \mp \vec{\beta}_i. \end{aligned} \quad (3.10)$$

This looks just like a conjugation by S of a T^{N_c} transformation. As we shall see below, this transformation agrees with the semi-classical monopole spectrum of the theory in the parts of moduli space for which the semi-classical spectrum should be self-dual. However, this is also true for many other transformations, so it is not clear that this is indeed the “correct” duality transformation. We might try to check if this duality transformation is consistent with the flow from $SU(N_c)$ gauge group to $SU(N_c - 1)$ gauge group, as described in [18]. If we naively compare the a ’s and a_D ’s of the two theories, we find that in fact this transformation is not consistent with the flow (we find for the $SU(N_c - 1)$ theory $B = N_c \tilde{C}_0^{-1}$ instead of $B = (N_c - 1) \tilde{C}_0^{-1}$). However, since the duality takes us to a strong coupling theory, the quantum corrections should be important, and we cannot trust this computation. For instance, for $G = SU(3)$, the flow takes place far from the surface \mathcal{M} in moduli space where (2.5) is correct quantum mechanically. Hence, we cannot really constrain the duality transformations by using this flow.

For $N_c = 3$, another possible duality transformation is $q_1 = q_4 = 0$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ up to Weyl transformations. This gives the duality transformation described in [22], taking τ to $-4/(3\tau)$. However, this transformation does not preserve the n -ality of charge vectors, and in any case for higher gauge groups we do not have analogs of this transformation. Therefore, it seems that $B = N_c C_0^{-1}$ is a more reasonable guess.

While it is possible to find consistent duality transformations for all N_c , it is not generally possible to find transformations whose square equals the identity (acting on the coupling constant τ), as seems to be implied from the transformations of the curves. This can easily be seen, for instance, for $N_c = 6$, which is one of the cases in which we can see from (3.7) that $\det(B)$ must be a multiple of $N_c^{N_c-2}$ (for odd N_c we can always find smaller solutions of (3.7)). In fact we can always write in this case $\det(B) = n^{N_c-1} N_c^{N_c-2}$ for an integer n . For the transformation squared to give unity (acting on τ), q_4 must equal $(-q_1)$, and then equation (3.7) becomes $(-q_1^2 - q_3)^{N_c-1} = 1/(n N_c)^{2(N_c-1)}$, and, therefore, $q_1^2 + q_3 = -1/(n N_c)^2$. Now, we can compute the determinant of C . Using this relation it

turns out to be $\det(C) = (-\frac{1}{(nN_c)^2} - q_1^2)^{N_c-1} N_c^{N_c}$, which must be an integer. Since N_c is not a rational number to the $(N_c - 1)$ 'th power for such N_c , this integer must necessarily be a multiple of N_c , leading to $(\frac{1}{(nN_c)^2} + q_1^2)N_c$ being an integer. Defining $r = nN_c q_1$, which must still of course be a rational number, we find that $r^2 + 1$ must be an integer multiple of $n^2 N_c$. For $N_c = 6$, taking this equation modulo 6, we find a contradiction, since there are no rational solutions modulo 6 to $r^2 + 1 = 0$. Thus, in this case there is no legal transformation which squares to unity. Note that if we use $B = N_c C_0^{-1}$, then for odd N_c we can choose $q_4 = -q_1 = \pm 1/N_c$ which gives a transformation that does square to unity, but this is not possible for even values of N_c .

Let us now briefly comment on the differences between this case and the $N = 4$ case, for which electric-magnetic duality is believed to work for all gauge groups, taking the gauge group to its dual by $\vec{\alpha}^i \leftrightarrow \vec{\beta}^i$. In this case, the electric charges are all in the root lattice of the gauge group, so that the definition of the $Sp(2r, \mathbf{Z})$ transformations above is changed. The matrix C appearing in the transformation of the magnetic to the electric charge numbers must now be a multiple of the Cartan matrix C_0 , which transforms the weights to the roots. The matrix B is allowed to be non-integer acting on the weights but must still be an integer when acting on the roots. Thus, in this case we can write $C = C_0 \tilde{C}$ and $BC_0 = \tilde{B}$ where \tilde{B}, \tilde{C} are integer valued matrices. Plugging this into equation (3.3) for $A = D = 0$ we find that necessarily $\tau \rightarrow b_0/\tau$ and $\tilde{C} = K^2 b_0 \tilde{B}$. In this case, however, we find $\det(\tilde{C}) \det(\tilde{B}) = (-1)^r$, so that $K^2 b_0 = -1$ and there are no problems in finding appropriate duality transformations. In particular, $\tilde{C} = -\tilde{B} = I$ and $A = D = 0$ gives the usual S duality transformation of the $N = 4$ theory, taking $\tau \rightarrow -4/\tau$ in our normalization of the coupling. Of course, in the $N = 4$ case there are no n -ality requirements analogous to those we discussed above, since all states have zero n -ality. We run into problems in the $N = 2$ case because in this case the electric charge lattice (which is the weight lattice of the gauge group) is generally not isomorphic to the magnetic charge lattice (which is the root lattice of the dual gauge group).

Our discussion so far assumed the validity of the classical relation (2.5). For the $N = 4$ and $G = SU(2)$ cases, this relation is valid quantum mechanically, hence our analysis is exact. In the $G = SU(3)$ case, the relation (2.5) is valid quantum mechanically along the surface \mathcal{M} , which is non-singular, and our analysis is still valid there. We have assumed that the surface \mathcal{M} is invariant under duality transformations, which is reasonable since it is the only surface along which the \mathbf{S}_3 Weyl symmetry is broken to \mathbf{Z}_3 . For higher gauge

groups there are no regions in the quantum moduli space for which (2.5) holds⁷, and our analysis is not valid in the quantum theory. Since the duality is a quantum effect, it is not clear that the classical part of the prepotential should be invariant by itself. However, since the transformations we are dealing with form a discrete group, it seems reasonable to expect that this will hold. In this case, the constraints that we have derived above will still be valid in the full quantum theory.

For $SU(N_c)$ gauge groups with $N_c > 3$ along the surface \mathcal{M} , there is a constant one-loop correction to the coupling matrix τ_{ij} . We can try to include this correction and repeat the analysis performed above. However, since this matrix consists of irrational numbers, we find that including this correction leaves no possible duality transformations, except for the trivial $\tau \rightarrow \tau + 2$ transformation. There are several possible ways to interpret this result. First, it is possible that for these gauge groups, along the surface \mathcal{M} , there are also non-perturbative corrections to the prepotential, and these must also be included. Second, it is possible that for $N_c > 3$ the duality does not preserve the surface \mathcal{M} , though it seems that this is the only surface along which the Weyl symmetry breaks to a \mathbf{Z}_n subgroup. Finally, it is possible that there is no exact S duality for these gauge groups. The first possibility seems to be the most reasonable one. For the scale invariant cases there are no rigorous arguments against the appearance of non-perturbative corrections, as long as these do not break the scale invariance. In the $SU(2)$ and $SU(3)$ cases, such corrections must have the same matrix form as the classical coupling matrix, since it is the only matrix invariant under the \mathbf{Z}_n Weyl transformations. Therefore, we can always swallow them in the definition of the coupling constant for strong coupling. For $N_c > 3$ other matrices may appear, and we need to understand better the non-perturbative corrections in order to perform the full quantum analysis of the duality.

4. The classical monopole spectrum

Next, we would like to compute the classical monopole spectrum of the $N = 2$ SQCD theories. We will show that classically there is an equivalence between BPS monopole solutions for real and complex Higgs fields, so that we may use old results [24,25] on the classical moduli space for real Higgs fields. By this we mean that for every BPS monopole solution for real Higgs fields there exists a solution for complex Higgs fields (with any VEV

⁷ As pointed out in a recent paper [32].

for the imaginary part of the Higgs field in the Cartan subalgebra), and for every solution for complex Higgs fields there is a solution for real Higgs fields. The dimension of the monopole moduli space, i.e. the number of bosonic zero modes, is, therefore, the same in both cases⁸.

Let us first derive the BPS bound for complex Higgs fields, and see what the BPS equations are in this case. The analysis will be purely classical throughout this section, and we will concentrate on solutions with no electric charge. We will take the action to be the bosonic part of the $N = 2$ SYM lagrangian,

$$S = \int d^4x \text{Tr} \left\{ -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(D_\mu \text{Re}(\Phi))^2 + \frac{1}{2}(D_\mu \text{Im}(\Phi))^2 - \frac{1}{2}([\text{Re}(\Phi), \text{Im}(\Phi)])^2 \right\}. \quad (4.1)$$

In (4.1) we have chosen, for simplicity, the theta angle to be zero, and we normalized the gauge fields so that the gauge coupling is one. The gluinos and the quark superfields are not expected to affect the monopole solutions, except via zero modes upon quantization. The energy density derived from this action is

$$\begin{aligned} U &= \frac{1}{2} \int d^3x \text{Tr} [(E_i)^2 + (B_i)^2 + (D_0 \text{Re}(\Phi))^2 + (D_i \text{Re}(\Phi))^2 + (D_0 \text{Im}(\Phi))^2 + \\ &\quad (D_i \text{Im}(\Phi))^2 + ([\text{Re}(\Phi), \text{Im}(\Phi)])^2] = \\ &= \frac{1}{2} \int d^3x \text{Tr} [(B_i - D_i \text{Re}(\Phi) + i D_i \text{Im}(\Phi))(B_i - D_i \text{Re}(\Phi) - i D_i \text{Im}(\Phi)) + \\ &\quad 2(B_i D_i \text{Re}(\Phi)) + ([\text{Re}(\Phi), \text{Im}(\Phi)])^2 + (E_i)^2 + (D_0 \text{Re}(\Phi))^2 + (D_0 \text{Im}(\Phi))^2]. \end{aligned} \quad (4.2)$$

All components of the energy density written above are clearly non-negative, except for $\int d^3x \text{Tr}(B_i D_i \text{Re}(\Phi))$, which is exactly v times the real part of the magnetic charge, $Q_M = v^{-1} \int d^3x \partial_i (\text{Tr}(B_i \Phi))$, where v is the absolute value of the asymptotic Higgs field. Therefore, we find that $U \geq v \text{Re}(Q_M)$, with equality only when all other parts of the energy density vanish. However, there is a chiral symmetry freedom which enables us to choose the phase of Φ (classically and also quantum mechanically when the beta function vanishes), and we derive the best bound by choosing the phase to make Q_M positive and real. Thus, generally we find that $U \geq v|Q_M|$. This is the BPS inequality for states with no electric charge, and we are looking for solutions to the classical equations of motion for which the inequality is saturated, i.e. $U = v|Q_M|$. From equation (4.2), we deduce

⁸ The exact relation between the two cases involves a Weyl transformation as described later in this section.

that if we choose the phase of the Higgs field as defined above, such solutions must satisfy (locally)

$$\begin{aligned}
B_i &= D_i \text{Re}(\Phi) \\
0 &= D_i \text{Im}(\Phi) \\
0 &= [\text{Re}(\Phi), \text{Im}(\Phi)] \\
E_i &= D_0 \text{Re}(\Phi) = D_0 \text{Im}(\Phi) = 0.
\end{aligned} \tag{4.3}$$

These equations generalize the relations $B_i = D_i \phi, E_i = D_0 \phi = 0$ which exist for a real Higgs field ϕ . Obviously, any solution (in the gauge $A_0^a = 0$) will be time independent, and we will assume this from here on. Relaxing the assumption of zero electric charge we would find that both B_i and E_i are proportional to $D_i \text{Re}(\Phi)$. Thus, the electric and magnetic charge vectors of classical BPS-saturated dyons are always proportional to each other.

Our goal in this section is to relate the solutions to (4.3) which exist for real and complex Higgs fields. We will look for local solutions, for which the asymptotic magnetic field decays as $1/r^2$. The solutions are then characterized by the asymptotic values of Φ and of the magnetic field along (for instance) the z axis. These asymptotic values commute, since $D_i \Phi$ vanishes asymptotically. Therefore, both of them may be chosen to be in the Cartan subalgebra. Following E. Weinberg [24] we will denote the asymptotic value of the Higgs field along the z axis by $\Phi_0 = v \vec{h} \cdot \vec{H}$, where \vec{H}_i is a basis for the Cartan subalgebra, and \vec{h} is a vector of unit absolute value. The asymptotic behavior of the magnetic field along the z axis will be of the form $B_z = \vec{g} \cdot \vec{H}/z^2$, where \vec{g} satisfies the quantization condition $\vec{g} = \sum_i n_m^i \frac{\vec{\alpha}^i}{(\vec{\alpha}^i)^2}$. The vectors \vec{g} and \vec{h} are defined by this procedure up to Weyl transformations, and the magnetic charge Q_M (defined above) is given by $Q_M = 4\pi \vec{g} \cdot \vec{h}$. In Weinberg's analysis [24] \vec{h} is real, and the Weyl transformation freedom is used to set $\vec{h} \cdot \vec{\alpha}^i \geq 0$ for all simple roots $\vec{\alpha}^i$. This removes any ambiguities for the case in which only an abelian symmetry remains unbroken, i.e. $\vec{h} \cdot \vec{\alpha} \neq 0$ for all roots $\vec{\alpha}$.

First, let us show that any classical solution for complex Higgs fields gives a solution for real Higgs fields with the same magnetic charge. This is obvious, since, as described above, we can always use a $U(1)_R$ transformation to choose the phase of the Higgs field in such a way that equation (4.3) is satisfied. This is achieved by choosing the phase of the vector \vec{h} above so that $\vec{g} \cdot \vec{h}$ is real and positive. Then, simply by setting $\text{Im}(\Phi) = 0$ we find a BPS solution for a real Higgs field with the same magnetic field, and with the same Higgs VEV as the real part (after the $U(1)_R$ rotation) of the original Higgs field.

The opposite direction is slightly more complicated. Let us assume that we are given a solution of the BPS condition for a real Higgs field. We will show that we can uniquely generate from it a solution for a complex Higgs field, with any expectation value in the Cartan subalgebra for the imaginary part of the Higgs field, perpendicular to the direction of the magnetic field. The last requirement is necessary to ensure that $Q_M > 0$. We need to generate an appropriate solution to equation (4.3), i.e. define a field $\text{Im}(\Phi)$ which equals the desired VEV far along the z axis (where we determined the asymptotic values) and satisfies

$$D_i \text{Im}(\Phi) = [\text{Re}(\Phi), \text{Im}(\Phi)] = 0. \quad (4.4)$$

The equation $D_i \text{Im}(\Phi) = \partial_i \text{Im}(\Phi) + [A_i, \text{Im}(\Phi)] = 0$ is of first order, and we can easily solve it for a given field A_i . The solution, given the value of $\text{Im}(\Phi)$ at a point x (which we choose far along the z axis), is given by

$$\text{Im}(\Phi)(y) = g_{xy} \text{Im}(\Phi)(x) g_{xy}^{-1} \quad (4.5)$$

where

$$g_{xy} = P \exp\left(-\int_y^x dx^i A_i\right). \quad (4.6)$$

We should now show that the solution (4.5) is uniquely defined (i.e. it does not depend on the path from x to y), and that it satisfies $[\text{Re}(\Phi), \text{Im}(\Phi)] = 0$ everywhere. Both properties can easily be shown to hold as a result of the equality $[B_i, \text{Im}(\Phi)] = 0$, which is satisfied by any solution to $D_i \text{Im}(\Phi) = 0$. First, any path which differs from the original path by a small loop in the definition of g_{xy} can easily be shown to give a solution differing by $[B_i, \text{Im}(\Phi)] = 0$. Second, we chose $\text{Im}(\Phi)$ so that at the point x $[\text{Re}(\Phi), \text{Im}(\Phi)] = 0$, and then the equations (4.3) lead to $D_i [\text{Re}(\Phi), \text{Im}(\Phi)] = [B_i, \text{Im}(\Phi)] = 0$. Therefore, any solution to $D_i \text{Im}(\Phi) = 0$ will automatically satisfy $[\text{Re}(\Phi), \text{Im}(\Phi)] = 0$ everywhere.

Thus, up to a choice of the VEV of the imaginary part of the Higgs field, there is a one-to-one correspondence between classical BPS monopole solutions for real Higgs fields and the classical BPS monopole solutions of the $N = 2$ theory. The former were analyzed by Weinberg [24], who found that, after eliminating the Weyl transformation freedom as described above, the dimension of the moduli space of solutions with magnetic charges n_m^i (for a given Higgs VEV) is

$$4 \sum_{i=1}^r n_m^i. \quad (4.7)$$

For example, for the embedding of the 't Hooft Polyakov monopole in the direction of a simple root of the dual gauge group, we find here just the three translational zero modes and the electric charge zero mode transforming the monopole into a dyon.

By the discussion above, the same result applies also to the $N = 2$ theory, after we use the $U(1)_R$ freedom to set $Q_M > 0$ and eliminate the Weyl freedom by choosing $\text{Re}(\vec{h}) \cdot \vec{\alpha}^i \geq 0$. However, it is important to notice that for a particular $\langle \phi \rangle$, we might need to perform different Weyl transformations for different monopoles in the spectrum of the theory, since we use different $U(1)_R$ transformations to set $Q_M > 0$. For complex $\langle \phi \rangle$ there is generally no choice of simple roots for which the number of zero modes is given by (4.7) for all monopoles, and we need to be careful when computing the number of zero modes of several monopoles for the same $\langle \phi \rangle$ (i.e. at the same point in moduli space).

The correspondence we found between real and complex $\langle \phi \rangle$ seems strange if we recall that Weinberg interpreted [24] this moduli space (for real Higgs field) by assuming the existence of fundamental monopoles with charges $\vec{\beta}^i$ ($i = 1, \dots, r$) which have no force between them, so that we can take them as far apart as we wish and still satisfy the BPS bound. Equation (4.7) then means that all monopoles of higher charges can be interpreted as bound states of the monopoles with charge $\vec{\beta}^i$. In the $N = 2$ theory this is no longer the case. There is no longer any way (for generic Higgs VEVs) to patch together monopole solutions in different directions (in the Cartan subalgebra), and most of the formerly marginally stable bound state solutions are now absolutely stable. However, we have assumed throughout the discussion a localized solution, with a $1/r^2$ falloff of the magnetic field. Clearly, a solution which consists of two monopoles far apart does not satisfy this conditions, and, therefore, cannot be generalized to complex fields as described above. Thus, there is no contradiction. Still, it would be interesting to understand how to interpret these classical “multi-monopole” solutions in the $N = 2$ theory.

There is a small subtlety which we have not yet addressed. Equation (4.7) for a real Higgs field is only correct when the real Higgs field completely breaks the gauge symmetry to $U(1)^r$. We assume throughout this paper that we are at a point at which the complex Higgs field completely breaks the gauge symmetry. However, on special surfaces in moduli space, which include the surface \mathcal{M} we were working on in the previous section, it could happen that the real part of the Higgs field (obtained after the $U(1)_R$ rotation as described above) does not by itself completely break the symmetry. This happens if, after the $U(1)_R$ rotation, $\text{Re}(\langle \phi \rangle) \cdot \vec{\alpha} = 0$ for some root $\vec{\alpha}$. Our correspondence between the solutions for real and complex Higgs fields still works, but we must now look at the computation for a

real Higgs field which does not completely break the gauge group. When the magnetic field has no component in the non-abelian part of the remaining gauge group, this computation was also performed by Weinberg in [25], with a result analogous to (4.7). However, in the other cases (which include the surface \mathcal{M}) the number of normalizable bosonic zero modes appears not to be completely known, and, therefore, we do not know the number of zero modes also in the $N = 2$ theory in these cases. Since for this to occur we need the projection of the Higgs field in the direction of two different roots to be real, this problem may arise only on singular surfaces on which states may be only marginally stable. Our discussion in the following sections will therefore be limited to generic points in the moduli space, where (4.7) is indeed correct.

5. The low-lying semi-classical monopoles

A general classical monopole solution has a moduli space of bosonic zero modes, and in the semi-classical computation we quantize the bosonic zero modes in this space. In general, this moduli space is a complicated hyper-Kähler surface [26], and the quantization of the bosonic zero modes is difficult, even for $SU(2)$ as in [16]. There are 4 bosonic zero modes which always exist, corresponding to translations (3 zero modes) and to time-dependent gauge transformations transforming the monopole into a dyon. When these are the only zero modes that exist, as for the $n_m = 1$ monopole for gauge group $SU(2)$, the semi-classical quantization of the bosonic zero modes is trivial, leading to a series of dyon states whose electric charges differ by multiples of the magnetic charge. Therefore, we start by checking when such a simple moduli space occurs.

From the analysis of the previous section, it is clear that such a moduli space may arise if, after the appropriate Weyl transformation, the magnetic charge of the monopole is exactly a simple root of the dual gauge group, $\vec{g} = \vec{\beta}^i$, so that the vector n_m^i is of the form $(0, 0, \dots, 0, 1, 0, \dots, 0)$. If $\langle \phi \rangle$ is real (up to a global phase), we use the same Weyl transformation for all monopoles, and then the monopoles corresponding to simple roots indeed have a four-dimensional moduli space. All other monopoles for which all n_m^i are positive have a larger moduli space. For complex $\langle \phi \rangle$ the situation is more complicated, and generally monopoles corresponding to any root of the dual group may be transformed by the Weyl transformation described above to simple roots, in which case their moduli space would be simple. A particular case which is easy to analyze is when $\langle \phi \rangle$ lies in the surface \mathcal{M} , defined by $u_i = 0$ for $i = 2, \dots, N_c - 1$. In this case, for $N_c \geq 3$, we can check and see

that no monopoles, including the simple root monopoles, are transformed by the relevant Weyl transformations to simple root monopoles. Thus, on this surface all monopoles have a moduli space of dimension larger than 4, whose quantization is complicated.

For general magnetic charges n_m^i (which may be positive or negative), equation (4.7), which determines the number of zero modes, is not necessarily positive. When the number is positive, the analysis shows that if the monopole exists, the dimension of its classical moduli space is (4.7), but it does not show that a solution indeed exists. The BPS mass formula (2.6) shows that for generic magnetic charges and at a generic point in moduli space, if a solution exists it is stable. The only generally known solutions when the gauge symmetry is broken to the abelian subgroup are the embeddings of the $SU(2)$ 't Hooft Polyakov monopole, which give monopoles whose magnetic charge is a root of the dual gauge group, and for these equation (4.7) is always a positive number. However, when (4.7) turns out to be negative or zero, it is obvious that no classical monopole solutions exist with these magnetic charges, since any solution must have at least the translation zero modes. For real $\langle\phi\rangle$, when no Weyl transformations are necessary in the analysis of the previous section, we can easily show that for every monopole whose charges n_m^i are not all non-negative (or all non-positive) there exist regions in the moduli space for which this monopole has a non-positive number of zero modes and, therefore, semi-classically it does not exist.

However, when we cross singular surfaces in moduli space, the number of zero modes (4.7) may change for a monopole which is only marginally stable on the surface, and classical monopole solutions which did not previously exist may arise. This phenomenon could occur also in the quantum theory [23]. Thus, monopoles having some positive and some negative n_m^i 's may exist in parts of the moduli space. Such monopoles may also have simple moduli spaces, like the simple root monopoles. For instance, in the $SU(3)$ theory, if a monopole of charges $n_m^i = (2, -1)$ exists near the surface of real $\langle\phi\rangle$ it has just 4 bosonic zero modes.

In fact, monopoles of this type must indeed exist in parts of the moduli space if the transformation (3.10) is indeed a symmetry of the theory. This transformation takes quarks whose electric charges are $\vec{e}_i = \vec{\mu}_i - \vec{\mu}_{i-1}$ (for $i = 1, \dots, N_c$, where we define $\vec{\mu}_0 = \vec{\mu}_{N_c} = 0$), which always exist in the theory for weak coupling, to states whose magnetic charges are proportional to \vec{e}_i . For these magnetic charges (for $i = 2, \dots, N_c - 1$) we can see, by the analysis of the previous section, that there exist parts of the moduli space (for instance, near the surface of real $\langle\phi\rangle$) for which the computation gives a non-positive number of zero

modes. Thus, for weak coupling these monopoles cannot exist there. For example, in the $SU(3)$ theory one of the quarks has an electric charge $\vec{\mu}_2 - \vec{\mu}_1$, which is transformed by (3.10) to magnetic charges $(1, -1)$. For these charges no solutions exist near real $\langle\phi\rangle$, when no Weyl transformations are involved in the analysis as described in the previous section. Obviously, if no classical solutions exist for these quantum numbers we do not find any such states in the spectrum by the semi-classical quantization. However, near the surface \mathcal{M} , defined by $u_i = 0$ ($i = 2, \dots, N_c - 1$), for which the spectrum should be self-dual, at least for $N_c = 3$, we find that these monopoles do always have a positive number of zero modes. They may, therefore, exist classically, though the corresponding classical solutions have not yet been built as far as we know. Thus, the classical analysis does not rule out the existence of these states in part of the moduli space (for weak coupling). It is easy to see that the monopoles related by (3.10) to the W bosons have a positive number of zero modes at any point in moduli space, since their magnetic charge is proportional to a root of the dual gauge group (recall that the roots transform among themselves under Weyl transformations). In fact, the number of bosonic zero modes we find for these monopoles is always at least $4N_c$.

Next, we would like to discuss the fermionic zero modes in the background of the monopole [27]. For any monopole, the gluinos have zero modes related by supersymmetry to the bosonic zero modes, which restore the $N = 2$ supersymmetry. In the absence of other fermionic zero modes, these zero modes turn the monopole into an $N = 2$ hypermultiplet [28]. The analysis of the quark zero modes is more complicated. Generally each quark flavor may have k zero modes in the background of the monopole field. For $SU(N_c)$ gauge groups we can combine the zero modes of the quark and the anti-quark into complex zero modes ρ_{iA} ($i = 1, \dots, N_f; A = 1, \dots, k$), which satisfy the anti-commutation relations $\{\rho_{iA}^\dagger, \rho_{jB}\} = \delta_{ij}\delta_{AB}$. The zero modes ρ_{iA} carry the same flavor quantum numbers as the quarks. They are in a fundamental representation of $SU(N_f)$ and carry a baryon number charge equal to that of the quarks. By acting with the zero modes on the monopole vacuum, we generate states in various (anti-symmetric) representations of the $SU(N_f)$ flavor group. However, in general, not all these states are BPS states. When the moduli space of the bosonic zero modes is non-trivial, only a small number of these states actually become BPS-saturated states when performing the semi-classical quantization. This is demonstrated in the analysis performed for $n_m = 2$ and gauge group $SU(2)$ in [16].

The number of zero modes of a quark in the fundamental (or anti-fundamental) representation in the background of a monopole of charges n_m^i may be determined by an index

theorem derived by Callias [29]. This is generally a laborious calculation. However, for the 't Hooft Polyakov embeddings, the problem is essentially reduced to an $SU(2)$ problem, and we may use the known results for $SU(2)$, computed explicitly in [29]. Note that for these monopoles a four-dimensional bosonic moduli space may arise, as discussed above, in which case all fermion zero modes indeed generate BPS saturated states. For these monopoles, the non-constant elements of the gauge fields A_i^a and of Φ^a are in a 2×2 block matrix, and thus only two elements of the fermion (regarded as a vector in the fundamental of $SU(N_c)$) can be non-zero. The VEV of the Higgs field in this 2×2 matrix may be divided into a part proportional to σ^3 , which acts as an $SU(2)$ Higgs VEV, and a part proportional to the identity matrix, which acts essentially as a (complex) mass term for the fermion. The analysis of [29] was performed with a real mass term, and it shows that a quark in the background of an $SU(2)$ monopole of $n_m = 1$, which is the monopole we are embedding, has one zero mode if the mass m lies between a and $-a$ (where the Higgs VEV is $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$), and no zero modes otherwise. When there is an imaginary part to the mass term, we should not actually look at zero modes of the Dirac equation, since the BPS formula in the presence of a mass shows (in the semi-classical limit) that the mass of the BPS saturated state with the quantum numbers of the monopole and fermion put together differs by $\text{Im}(m)$ from that of the monopole. Thus, in this case we should look at solutions to the Dirac equation whose energy is $\text{Im}(m)$, but this is exactly the energy that we naturally find for the zero-mode solution of real mass in the presence of the complex mass term. Hence, the imaginary part of the mass term does not affect the analysis, and we find that a quark has a zero mode in the background of a 't Hooft Polyakov embedding with magnetic charge $\vec{g} = \vec{\beta}^i + \vec{\beta}^{i+1} + \dots + \vec{\beta}^j$ if (and only if)

$$|\text{Re}(\langle\phi\rangle) \cdot (\vec{\mu}_i - \vec{\mu}_{i-1} - \vec{\mu}_j + \vec{\mu}_{j+1})| \leq |\langle\phi\rangle \cdot \vec{g}| \quad (5.1)$$

(where we choose the phase of $\langle\phi\rangle$ so that $\langle\phi\rangle \cdot \vec{g}$ is real).

It is easy to show that for real (or almost real) $\langle\phi\rangle$ exactly one of the monopoles of charge $\vec{\beta}^i$, which have a simple bosonic moduli space, has a zero mode. For general complex $\langle\phi\rangle$ more complicated situations may arise, and in particular near the surface \mathcal{M} it may easily be seen that all the monopoles corresponding to roots of the dual gauge group (i.e. the 't Hooft Polyakov embeddings) have quark zero modes.

6. Duality of the spectrum of $N = 2$ SQCD

We would like now to bring together all of our results, and see if we can find a consistent electric-magnetic duality transformation for the $N = 2$ SQCD theory. For all electric and magnetic charges, we should find the same flavor quantum numbers for the states which transform into each other under the duality, up to a possible automorphism of the $SU(2N_c) \times U(1)_B$ flavor group.

In order to compare states related by the duality we should generally compare states at weak coupling with states at strong coupling. However, since our methods for computing the particle spectrum are semi-classical, we can only compute the spectrum at weak coupling. Thus, such a comparison is generally impossible, unless we can connect the spectrum at weak coupling with the spectrum at strong coupling. For $G = SU(2)$, for the $N = 4$ SYM theories and for $G = SU(3)$ along the surface \mathcal{M} , we can see from (2.5) that we do not cross any singular surfaces in moving from strong to weak coupling. Hence, the particle spectrum cannot change. Along this surface, which we assume to be transformed to itself by the duality, the spectrum found at weak coupling by the semi-classical computation should be self-dual. In other parts of the moduli space, we cannot discard the possibility that the spectrum changes at some value of τ , and we cannot generally make such a comparison. In fact, we will see that in other parts of the moduli space the semi-classical spectrum is not self-dual, and such phase transitions changing the spectrum of the theory apparently do indeed occur as we change the value of τ .

There are some states of the theory which always exist in the semi-classical region. These are the quark hypermultiplets in the fundamental of $SU(N_f)$ with baryon number $B = 1$, and the W bosons which reside in vector multiplets and are singlets of the flavor group. These representations should be identical to those we find for the monopole states connected with the quark and W -boson states by duality, up to a possible automorphism of the flavor group.

As discussed in the previous section, the monopole generally gets flavor quantum numbers due to the zero modes of the quarks, which exist in the background of the monopole [27], acting on the monopole state. Acting with the zero modes may also generally change the electric charge of the monopole state. For $SU(2)$, there exists a simple projection determining which flavor states may actually have $n_e^i = 0$, given by equation (5.2) of [5]. This equation relates the charge of the state under the center of the gauge group, which is naturally determined by the electric charge of the state, with the charge of the state under

the center of the flavor group. However, we do not know how to generalize this equation to a general gauge group, since the gauge transformation used to derive this equation (a rotation around the direction of the Higgs field [30]) is generically not in the center of the gauge group.

In the $SU(2)$ case, Seiberg and Witten argue (at the end of section 5 of [5]) that the center of the flavor group is always faithfully represented on the lattice of charges, i.e. that the charge of a state uniquely determines the charge of the flavor representation of that state under the center of the flavor group. However, it is not clear whether this should be true in general. In fact, we can easily see that it is not true even for the elementary quark states in the $SU(N_c)$ case. The center of the flavor group for $N_c > 2$ includes a \mathbf{Z}_{2N_c} group, and the sum of the electric charges of $N_c - 1$ different quark states (whose \mathbf{Z}_{2N_c} flavor group charge is 1) gives the electric charge of an anti-quark state (whose \mathbf{Z}_{2N_c} charge is -1). However, for the elementary electric states the gauge group \mathbf{Z}_{N_c} charge equals (modulo N_c) the flavor group \mathbf{Z}_{2N_c} charge. Therefore, it seems possible to have a relation between the flavor n -ality of a state (modulo N_c) and its electric charge. This relation certainly exists for the elementary electric states. It is satisfied also by the monopole states if the color n -ality of the monopole vacuum, on which the quark zero modes act, is zero, which seems to be the case.

The electric charge of the quark zero modes is not well defined in general, but their n -ality (the \mathbf{Z}_{N_c} charge) is well defined. Thus, we expect that states generated by acting with k quark zero modes on the monopole vacuum will have an n -ality which is larger by k than that of the monopole vacuum. For a 't Hooft Polyakov embedding of magnetic charge $\vec{g} = \vec{\beta}^i + \vec{\beta}^{i+1} + \dots + \vec{\beta}^j$, the only non-zero color components of the quark zero modes have electric charges $\vec{\mu}_i - \vec{\mu}_{i-1}$ and $\vec{\mu}_{j+1} - \vec{\mu}_j$. We expect, therefore, the states generated by the zero modes to differ by these charge vectors. For these monopoles with magnetic charge \vec{g} it does not matter which of the two charge vectors we take, since the difference between them is \vec{g} , and the T^2 transformation relates states differing by electric charge \vec{g} .

Let us now discuss how all this affects the duality of the spectrum. We will discuss in detail only the gauge group $SU(3)$ case, though most of the discussion may be easily generalized. As discussed above, the only region where we can easily see that the particle spectrum must be self-dual is near the surface \mathcal{M} with $u_2 = 0$. Along this surface, to the extent that we have computed it, we find that the transformation (3.10) is consistent with the semi-classical spectrum. All of the magnetic charges related to quarks and W bosons by this transformation give a positive number of zero modes by (4.7), including the

$n_m^i = (1, -1)$ monopole. The resulting monopoles all have a non-trivial monopole moduli space (with at least 12 bosonic zero modes), and may also have fermionic zero modes that would give precisely the correct flavor quantum numbers for these states to be related to the electric states. The 't Hooft Polyakov embeddings also have complicated moduli spaces (with 8 bosonic zero modes) in the vicinity of \mathcal{M} , and for all of them the quarks have zero modes. Since the bosonic moduli space is complicated for all of the monopoles, it is not clear which flavor representations actually exist as BPS saturated states generated from these monopoles, and at what electric charges. The “electric-magnetic” duality of (3.10) transforms some of these states among themselves, and looks like it could be consistent with the semi-classical spectrum. A more comprehensive analysis is necessary, however, to verify that (at least near the surface \mathcal{M}) the spectrum is indeed self-dual.

Next, let us analyze the semi-classical particle spectrum for real (or close to real) $\langle\phi\rangle$ ($4u_2^3/27u_3^2 > 1$ in gauge invariant variables), and check if it is self-dual. In this case, as discussed above, both monopoles corresponding to the simple roots $\vec{\beta}^i$ of the dual gauge group have a simple bosonic moduli space, and one of them has a fermionic zero mode while the other does not. For the monopole which has a fermionic zero mode, we act with the quark zero modes to get states in the $\mathbf{1} + \mathbf{6} + \mathbf{15} + \mathbf{20} + \overline{\mathbf{15}} + \overline{\mathbf{6}} + \mathbf{1}$ representations of the $SU(6)$ flavor group, with rising baryon numbers from left to right. It is not clear how to determine the absolute baryon number of the monopole state, because we can always add to the baryon number any linear combination of conserved charges. Therefore, we will ignore the baryon number in our analysis. If this monopole is related by duality to a quark state (as for $SU(2)$), we expect to find just a $\mathbf{6}$ representation (or a $\overline{\mathbf{6}}$ if there is an outer automorphism of the $SU(6)$ flavor group involved). In section 3 we showed that states related by duality to quarks always have an electric charge vector proportional to the magnetic charge vector. For \vec{g} corresponding to a root vector, this means that their electric charge must be a multiple of \vec{g} , which must be zero (up to T^2 transformations) for it to lie in the electric charge lattice. It is not apriori clear which of the states generated by the quark zero modes has zero electric charge, since we do not know the electric charge of the monopole vacuum. However, at least for real $\langle\phi\rangle$ there seems to be (classically) a CP symmetry in this theory (as described in [31] for the $SU(2)$ case) inverting the electric charges but not the magnetic charges (for $\theta = 0$). In this case, obviously, the state in the $\mathbf{20}$ representation has zero electric charge. Thus, if the semi-classical analysis is relevant in this part of moduli space, then this monopole cannot be related to a quark by duality. In fact, by similar arguments we can show that the monopoles generated by

't Hooft Polyakov embeddings, whose magnetic charges are roots of the dual gauge group, can never be related to the quarks by duality for $N_c > 2$.

The monopole corresponding to the other simple root has no quark zero modes in this case, and it is, therefore, a flavor singlet which resides (after using the gluino zero modes) in an $N = 2$ hypermultiplet. There is no similar state with only electric charges. Hence, this monopole also cannot transform into purely electric states. In any case it seems that, if the semi-classical analysis is relevant, the quarks are not connected to the “fundamental” monopoles of charge $\vec{\beta}^i$ by a duality transformation.

The quarks could still be related to monopoles of higher magnetic charge, as in (3.10). The semi-classical quantization of these monopoles is much more difficult. However, as discussed in the previous section, this transformation takes the quark state with electric charge $\vec{e} = \vec{\mu}_2 - \vec{\mu}_1$ to a monopole of charges $n_m^i = (1, -1)$ which does not exist near the surface of real $\langle\phi\rangle$, and obviously it cannot be the correct transformation there. Thus, it seems that the semi-classical particle spectrum near the surface of real $\langle\phi\rangle$ is not self-dual. As discussed above, in this area of moduli space we cannot rule out changes in the spectrum as we go from strong to weak coupling, and the semi-classical spectrum does not have to be self-dual for the duality to hold. Our discussion shows that if the duality holds, then the spectrum necessarily changes as we move from strong to weak coupling near the surface of real $\langle\phi\rangle$.

The situation in all $SU(N_c)$ gauge theories is actually similar to the situation described above for the $SU(3)$ theory. For all N_c , we find near the surface \mathcal{M} that all of the monopoles related by (3.10) to the quarks may indeed exist, and always have many bosonic zero modes, making their semi-classical quantization complicated. The 't Hooft Polyakov embeddings all have at least 8 bosonic zero modes near this surface. Thus, we cannot even tell which BPS states are generated from these monopoles near this surface without performing complicated computations. The duality seems to be consistent along the surface \mathcal{M} , even though we have not been able to show that this must be the case for $N_c > 3$. However, many possible transformations are consistent with the spectrum at the classical level, and it appears to be necessary to perform (at least) the full semi-classical calculations (as in [16]) in order to really check the duality. For all $SU(N_c)$ groups ($N_c > 2$) we find that the particle spectrum for real $\langle\phi\rangle$ is not self-dual. Therefore, if the duality holds, the spectrum in this region must change as we move between strong and weak coupling.

7. $N = 2$ SQCD with non-zero quark masses

In our discussion up to now we focused on the case of zero quark masses. In fact, it seems that all tests of S duality so far, except for the duality of the low energy spectrum-generating curve, have focussed on the massless case (in the $N = 4$ theory they were performed without a mass term for the adjoint field which breaks the supersymmetry to $N = 2$). However, apriori we see no reason why exact electric-magnetic duality may not also be true in theories with quark masses which transform appropriately under the duality.

Thus, we would like to try and test the duality by semi-classical computations also in the theories with massive quarks. Note that mass terms explicitly break the scale invariance and $U(1)_R$ symmetries that we used in the classical analysis of sections 4 and 5, and the analysis has to be changed appropriately. However, this analysis runs into the same problems that we ran into in the previous sections for cases other than $G = SU(3)$ along the surface \mathcal{M} . Again, we cannot prove that when we go from strong to weak coupling we do not cross singular surfaces, changing the particle spectrum of the theory. In fact, this is true even for the $SU(2)$ theory of [5]. For instance, if we look at this theory for 4 equal quark masses m and for weak coupling⁹, then we know that for $u \ll m^2$ the theory looks like an $N_f = 0$ theory, with a dynamically generated scale Λ proportional to $mq^{1/4}$ (where $q = e^{i\pi\tau}$). In this theory we know [4] that there are two singularities, at $u = \pm\Lambda^2$, there is a marginal stability curve passing through both of them, and the particle spectrum of the theory changes when we cross it. Since Λ depends on τ , it is clear that this curve moves as we change τ , so that the spectrum for the same u at strong and weak coupling is not necessarily the same.

As in the previous section, we can see that the semi-classical spectrum of this theory is not invariant under the S duality transformation of [5], signifying that such a change in the spectrum does indeed occur. The S duality of [5] transforms the theory with masses $m_i = (m, m, m, m)$ to a theory with masses $m_i = (2m, 0, 0, 0)$. The $SU(4)$ symmetry acting on the four quarks is transformed to the $SO(6) \sim SU(4)$ symmetry acting on the last three massless quarks. The $SU(4)$ quantum numbers of states related by the duality should be the same. In the theory with masses $m_i = (m, m, m, m)$, when $0 < u < m^2$, the quarks have no zero modes in the background of the $n_m = 1$ monopole. Therefore, semi-classically there are no states in the **6** representation of the $SU(4)$ flavor group, because these may only be generated by quark zero modes. However, in the theory with masses

⁹ We thank M. R. Plesser for this example.

$m_i = (2m, 0, 0, 0)$, the three massless quarks are in the **6** representation of $SO(6) \sim SU(4)$, and these obviously exist at weak coupling for all u . Thus, at least in part of the moduli space, the spectrum must indeed change from strong to weak coupling for the duality to hold.

This phenomenon is completely general. For instance, for general N_c , a quark with mass m_i becomes marginally stable when $\text{Im}(\frac{a_1 + m_i/\sqrt{2}}{a_D}) = 0$. The form of this curve in moduli space changes when we change the coupling τ , on which both a and a_D depend since they are periods of a curve depending on τ . This is true also for the $N = 2$ theory with a massive hypermultiplet, obtained by a mass perturbation from the $N = 4$ theory.

Therefore, in general, we cannot check the duality in these theories by requiring that the semi-classical spectrum should be self-dual, up to the transformation of the masses. For small masses (relative to $\langle\phi\rangle$), it is clear, since the curves all depend continuously on the masses, that near \mathcal{M} the spectrum should still be self-dual, since, in this case, we do not expect to cross surfaces along which the low-lying states are marginally stable. The monopole spectrum is self-dual in this case just as in the zero mass case, since when the masses are small the fermions still have the same zero modes. The quark-mass corrections to the BPS formula for the monopole mass are negligible for small masses at weak coupling. However, without computing the explicit form of the singular surfaces, we cannot check the duality by semi-classical computations in the general massive case.

8. Summary and conclusions

In this paper we analyzed $N = 2$ SQCD theories with zero beta function, in an attempt to understand their duality transformations and check them by a semi-classical analysis. We found that a priori many transformations in the $Sp(2r, \mathbf{Z})$ low-energy symmetry group may be exact symmetries of the theory. Unlike the $N = 4$ and $SU(2)$ cases, the transformation $\tau \rightarrow -1/\tau$ is not a symmetry in the general case. Clearly, more constraints are needed to find what is the exact duality group of the theory. Unfortunately, the semi-classical analysis does not provide many constraints on the duality in these theories, since in general the particle spectrum at weak coupling and at strong coupling need not be the same. The only part of moduli space for which the spectra must be the same is the surface \mathcal{M} for the case of $G = SU(3)$ with zero quark masses. In this case we found that the computations involved in the semi-classical quantization are difficult even for the monopoles of small n_m^i , and we did not perform them. The only test we can do

without performing either the computations on the moduli space of these monopoles, or the computation of the singular surfaces from the spectrum-generating curve, is to check that classical solutions may exist with appropriate magnetic charges to be related by duality to the quarks and W bosons. This is indeed the case, though these classical solutions have not yet been constructed. For higher $SU(N_c)$ groups we also found the duality to be consistent with our classical computations along the surface \mathcal{M} . Quantum corrections appear to be important for $N_c > 3$, as discussed at the end of section 3, and these theories deserve further investigations.

Our discussion was limited to $SU(N_c)$ gauge groups, but all of it can be straightforwardly generalized to any gauge group. For other gauge groups the calculation of section 3 will give different constraints, and perhaps simple semi-classical computations may give more constraints on the duality there, provided that the analysis of the particle spectrum along the surface analogous to \mathcal{M} is simpler.

Obviously, more tests should be made to verify the existence of an exact electric-magnetic duality transformation in these theories, and to find its exact form. One possible direction is to make better semi-classical computations, by generalizing the computations of [16] to the monopole moduli spaces which we find in these theories. Another direction is to compute the singular surfaces of these theories, in order to find the exact limits of validity of the semi-classical analysis, i.e. the values of the u_i for which a certain state does not become marginally stable when going from weak to strong coupling. Other directions for verifying the duality may involve computing various partition functions of these theories, perhaps after twisting them to topological theories.

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References

- [1] C. Montonen and D. Olive, “Magnetic monopoles as gauge particles ?”, Phys. Lett. **72B** (1977) 117; P. Goddard, J. Nuyts, and D. Olive, “Gauge theories and magnetic charge”, Nucl. Phys. **B125** (1977) 1
- [2] H. Osborn, “Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1”, Phys. Lett. **83B** (1979) 321
- [3] P. Howe, K. Stelle and P. West, “A class of finite four-dimensional supersymmetric field theories”, Phys. Lett. **124B** (1983) 55
- [4] N. Seiberg and E. Witten, “Monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory”, hep-th/9407087, Nucl. Phys. **B426** (1994) 19
- [5] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD”, hep-th/9408099, Nucl. Phys. **B431** (1994) 484
- [6] N. Seiberg, “Electric-magnetic duality in supersymmetric nonabelian gauge theories”, hep-th/9411149, Nucl. Phys. **B435** (1995) 129
- [7] R. G. Leigh and M. J. Strassler, “Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory”, hep-th/9503121, Nucl. Phys. **B447** (1995) 95
- [8] F. A. Bais, “Charge-monopole duality in spontaneously broken gauge theories”, Phys. Rev. **D18** (1978) 1206
- [9] R. Donagi and E. Witten, “Supersymmetric Yang-Mills theory and integrable systems”, hep-th/9510101, IASSNS-HEP-95-78
- [10] C. Vafa and E. Witten, “A strong coupling test of S duality”, hep-th/9408074, Nucl. Phys. **B431** (1994) 3
- [11] L. Girardello, A. Giveon, M. Porrati and A. Zaffaroni, “Nonabelian strong-weak coupling duality in (string derived) $N = 4$ supersymmetric Yang-Mills theories”, hep-th/9406128, Phys. Lett. **334B** (1994) 331; “S duality in $N = 4$ Yang-Mills theories with general gauge groups”, hep-th/9502057, Nucl. Phys. **B448** (1995) 127
- [12] A. Sen, “Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $SL(2, Z)$ invariance in string theory”, hep-th/9402032, Phys. Lett. **329B** (1994) 217
- [13] M. Porrati, “On the existence of states saturating the Bogomol’nyi bound in $N = 4$ supersymmetry”, hep-th/9505187, NYU-TH-95-05-03
- [14] P. C. Argyres and M. R. Douglas, “New phenomena in $SU(3)$ supersymmetric gauge theory”, hep-th/9505062, Nucl. Phys. **B448** (1995) 93
- [15] P. C. Argyres, M. R. Plesser, N. Seiberg and E. Witten, “New $N = 2$ superconformal field theories in four dimensions”, hep-th/9511154, RU-95-81, WIS-95-59, IASSNS-HEP-95/95

- [16] J. P. Gauntlett and J. A. Harvey, “ S -duality and the dyon spectrum in $N = 2$ super Yang-Mills theory”, hep-th/9508156, EFI-95-56; S. Sethi, M. Stern and E. Zaslow, “Monopole and dyon bound states in $N = 2$ supersymmetric Yang-Mills theories”, hep-th/9508117, HUTP-95-A031
- [17] A. Hanany and Y. Oz, “On the quantum moduli space of vacua of $N = 2$ supersymmetric $SU(N_c)$ theories”, hep-th/9505075, Nucl. Phys. **B452** (1995) 283
- [18] P. C. Argyres, M. R. Plesser and A. D. Shapere, “The Coulomb phase of $N = 2$ supersymmetric QCD”, hep-th/9505100, Phys. Rev. Lett. **75** (1995) 1699
- [19] P. C. Argyres and A. D. Shapere, “The vacuum structure of $N = 2$ super-QCD with classical gauge groups”, hep-th/9509175, RU-95-61
- [20] A. Hanany, “On the quantum moduli space of vacua of $N = 2$ supersymmetric gauge theories”, hep-th/9509176, IASSNS-HEP-95-76
- [21] N. Seiberg, “Supersymmetry and nonperturbative beta functions”, Phys. Lett. **206B** (1988) 75
- [22] J. A. Minahan and D. Nemeschansky, “Hyperelliptic curves for supersymmetric Yang-Mills”, hep-th/9507032, USC-95-019
- [23] S. Cecotti, P. Fendley, K. Intriligator and C. Vafa, “A new supersymmetric index”, hep-th/9204102, Nucl. Phys. **B386** (1992) 405; S. Cecotti and C. Vafa, “On classification of $N = 2$ supersymmetric theories”, hep-th/9211097, Comm. Math. Phys. **158** (1993) 569; M. Henningson, “Discontinuous BPS spectra in $N = 2$ gauge theory”, hep-th/9510138, YCTP-P14-95
- [24] E. Weinberg, “Fundamental monopoles and multimonopole solutions for arbitrary simple gauge groups”, Nucl. Phys. **B167** (1980) 500
- [25] E. Weinberg, “Fundamental monopoles in theories with arbitrary symmetry breaking”, Nucl. Phys. **B203** (1982) 445
- [26] M. F. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press (1988)
- [27] R. Jackiw and C. Rebbi, “Solitons with fermion number $1/2$ ”, Phys. Rev. **D13** (1976) 3398
- [28] J. P. Gauntlett, “Low energy dynamics of $N = 2$ supersymmetric monopoles”, hep-th/9305068, Nucl. Phys. **B411** (1994) 443
- [29] C. Callias, “Index theorems on open spaces”, Comm. Math. Phys. **62** (1978) 213
- [30] E. Witten, “Dyons of charge $e\theta/2\pi$ ”, Phys. Lett. **86B** (1979) 283
- [31] N. S. Manton and B. Schroers, “Bundles over moduli spaces and the quantization of BPS monopoles”, Ann. Phys. **225** (1993) 290
- [32] J. A. Minahan and D. Nemeschansky, “ $N = 2$ super Yang-Mills and subgroups of $SL(2, \mathbf{Z})$ ”, hep-th/9601059, USC-96-01